Some problems on cycle-distributed graphs

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Outline

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Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, Erdős raised the problem of determining $f(n)$ (see J.A. Bondy and U.S.R. Murty [Graph Theory with Applications (Macmillan, New York, 1976).], p.247, Problem 11). Y. Shi[On maximum cycle-distributed graphs, Discrete Math. 71(1988) 57-71.] proved that

Theorem 1.1 (Shi 1988)

$$f(n) \geq n + [(\sqrt{8n-23}+1)/2]$$

for $n \geq 3$.

Y. Shi[The number of edges in a maximum cycle distributed graph, Discrete Mathematics, 104(1992), 205-209.] , Y. Shi [On simple MCD graphs containing a subgraph homemorphic to K_4, Discrete Math , 126(1994), 325-338.], X. Jia[Some extremal problems on cycle distributed graphs, Congr. Number., 121(1996), 216-222.],

G. Chen, J. Lehel, M. S.Jacobson, and W. E.Shreve [Note on graphs without repeated cycle lengths, J. Graph Theory, 29(1998),11-15.], C. Lai[On the maximum number of edges in a graph in which no two cycles have the same length. Combinatorics, graph theory, algorithms and applications (Beijing, 1993), 447–450, World Sci. Publ., River Edge, NJ, 1994.], C. Lai[A lower bound for the number of edges in a graph containing no two cycles of the same length, Electron. J. of Combin. 8(2001), #N9, 1 - 6.], S. Liu[Some extremal problems on the cycle length distribution of graphs. J. Combin. Math. Combin. Comput. 100 (2017), 155–171.] obtained some results.

E. Boros, Y. Caro, Z. Füredi and R. Yuster[Covering non-uniform hypergraphs, J. Combin. Theory Ser. B 82(2001), 270-284.] proved that

Theorem 1.2 (Boros, Caro, Füredi and Yuster 2001)

$$f(n) \leq n + 1.98\sqrt{n}(1 + o(1)).$$

C. Lai [On the size of graphs without repeated cycle lengths, Discrete Appl. Math. 232 (2017), 226-229.] improved the lower bound.

Theorem 1.3 (Lai 2017) Let $t = 1260r + 169 \ (r \geq 1)$, then

$$f(n) \geq n + \frac{107}{3}t + \frac{7}{3}$$

for $n \geq \frac{2119}{4}t^2 + 87978t + \frac{15957}{4}$.

C. Lai[On the size of graphs with all cycle having distinct length, Discrete Math. 122(1993) 363-364.] proposed the following conjecture:

Conjecture 1.4 (Lai 1993)

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \leq \sqrt{3}.$$

It would be nice to prove that

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \leq \sqrt{3+\frac{3}{5}}.$$

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \leq \sqrt{3+\frac{1}{3}}.$$

C. Lai [On the number of edges in some graphs, Discrete Applied Mathematics 283 (2020), 751-755] construct a graph G having no two cycles with the same length which leads to the following result.

Theorem 1.5 (Lai 2020) Let $t = 1260r + 169$ $(r \geq 1)$, then

$$f(n) \geq n + \frac{119}{3}t - \frac{26399}{3}$$

for $n \geq \frac{1309}{2}t^2 - \frac{1349159}{6}t + \frac{6932215}{3}$.

From Theorem 1.5, we have

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{40}{99}},$$

which is better than the previous bounds $\sqrt{2}$ (see Shi 1988), $\sqrt{2+\frac{7654}{19071}}$ (see Lai 2017).

Combining this with Boros, Caro, Füredi and Yuster's upper bound, we get

$$1.98 \geq \limsup_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{40}{99}}.$$

The sequence $(c_1, c_2, \cdots, c_n)$ is the cycle length distribution of a graph G of order n, where c_i is the number of cycles of length i in G. Let $f(a_1, a_2, \cdots, a_n)$ denote the maximum possible number of edges which satisfies $c_i \leq a_i$ where a_i is a nonnegative integer. Y. Shi posed the problem of determining $f(a_1, a_2, \cdots, a_n)$, which extended the problem due to Erdős, it is clearly that $f(n) = f(1, 1, \cdots, 1)$ (see [Y. Shi, Some problems of cycle length distribution, J. Nanjing Univ. (Natural Sciences), Special Issue On Graph Theory, 27(1991), 233-234]).

Let $g(n,m) = f(a_1, a_2, \cdots, a_n)$, $a_i = 1$ for all i/m be integer, $a_i = 0$ for all i/m be not integer. It is clearly that $f(n) = g(n, 1)$.

From the Theorem 1.5, we have

$$\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{\frac{n}{m}}} \geq \sqrt{2+\frac{40}{99}},$$

for all integer m.

We obtain the following result.

Theorem 1.6 (Lai 2020) Let m be even, $s_1 > s_2, s_1 + 3s_2 > k$, then

$$g(n,m) \geq n + (k + s_1 + 2s_2 + 1)t - 1$$

for
$n \geq (\frac{3}{4}mk^2 + \frac{1}{2}mks_1 + \frac{3}{2}mks_2 + \frac{1}{2}ms_1^2 + \frac{3}{2}ms_1s_2 + \frac{9}{4}ms_2^2 + mk + ms_1 + 3ms_2 + \frac{1}{2}m)t^2 + (\frac{1}{4}mk + \frac{1}{2}ms_1 + \frac{3}{4}ms_2 - k - s_1 - 2s_2 + \frac{1}{2}m - 1)t + 1.$

From Theorem 1.6, we have

$$\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{\frac{n}{m}}} \geq$$

$$\sqrt{\frac{(k+s_1+2s_2+1)^2}{(\frac{3}{4}k^2+\frac{1}{2}ks_1+\frac{3}{2}ks_2+\frac{1}{2}s_1^2+\frac{3}{2}s_1s_2+\frac{9}{4}s_2^2+k+s_1+3s_2+\frac{1}{2})}},$$

for all even integer m.

Let $s_1 = 28499066, s_2 = 4749839, k = 14249542$, then

$$\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{\frac{n}{m}}} > \sqrt{2.444},$$

for all even integer m.

We make the following conjecture:

Conjecture 1.7 (Lai 2020)

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} > \sqrt{2.444}.$$

Proof of Theorem 1.6

Let $n_t = (\frac{3}{4}mk^2 + \frac{1}{2}mks_1 + \frac{3}{2}mks_2 + \frac{1}{2}ms_1^2 + \frac{3}{2}ms_1s_2 + \frac{9}{4}ms_2^2 + mk + ms_1 + 3ms_2 + \frac{1}{2}m)t^2 + (\frac{1}{4}mk + \frac{1}{2}ms_1 + \frac{3}{4}ms_2 - k - s_1 - 2s_2 + \frac{1}{2}m - 1)t + 1$, m be even, $s_1 > s_2, s_1 + 3s_2 > k$, $n \geq n_t$. It suffice to show that there exists a graph G on n vertices with $n + (k + s_1 + 2s_2 + 1)t - 1$ edges such that all cycles in G have distinct lengths and all the lengths of cycles are the multiple of m.

Now we construct the graph G which consists of a number of subgraphs: B_i, $(0 \leq i \leq s_1 t, i = s_1 t + j \ (1 \leq j \leq s_2 t), i = s_1 t + s_2 t + j \ (1 \leq j \leq t))$.

Now we define these $B_i s$. These subgraphs all only have a common vertex x, otherwise their vertex sets are pairwise disjoint.

For $1 \leq i \leq s_2t$, let the subgraph B_{s_1t+i} consists of a cycle

$$xa_i^1a_i^2...a_i^{ms_1t+2ms_2t+mi-1}x$$

and a path:

$$xa_{i,1}^1a_{i,1}^2...a_{i,1}^{\frac{ms_1t-ms_2t+mi}{2}-1}a_i^{\frac{ms_1t+ms_2t+mi}{2}}.$$

Based on the construction, B_{s_1t+i} contains exactly three cycles of lengths:

$$ms_1t + mi, ms_1t + ms_2t + mi, ms_1t + 2ms_2t + mi.$$

For $1 \leq i \leq t$, let the subgraph $B_{s_1t+s_2t+i}$ consists of a cycle

$$C_{s_1t+s_2t+i} = xy_i^1y_i^2...y_i^{ms_1t+3ms_2t+mk(k+1)t+mi-1}x$$

and k paths sharing a common vertex x, the other end vertices are on the cycle $C_{s_1t+s_2t+i}$:

$$xy_{i,p}^{1}y_{i,p}^{2}\ldots y_{i,p}^{\frac{ms_1t+3ms_2t-mkt+m(p-1)t+mi}{2}-1}y_{i}^{\frac{ms_1t+3ms_2t+mk(2p-1)t+m(p-1)t+mi}{2}}$$

$$(p=1,2,\ldots,k).$$

As a cycle with k chords contains $\binom{k+2}{2}$ distinct cycles, $B_{s_1t+s_2t+i}$ contains exactly $\frac{(k+2)(k+1)}{2}$ cycles of lengths:

$$ms_1t + 3ms_2t + mkht + (h + j - 1)mt + mi$$

$$(j \geq 1, h \geq 0, k + 1 \geq j + h).$$

B_0 is a path with an end vertex x and length $n - n_t$. The other B_i is simply a cycle of length mi.

Then $g(n,m) \geq n + (k + s_1 + 2s_2 + 1)t - 1$, for $n \geq n_t$.

This completes the proof.

Proof of Theorem 1.5

Let $n_t = \frac{1309}{2}t^2 - \frac{1349159}{6}t + \frac{6932215}{3}$, $t = 1260r + 169, r \geq 1$, $n \geq n_t$. It suffice to show that there exists a graph G on n vertices with $n + \frac{119}{3}t - \frac{26399}{3}$ edges such that all cycles in G have distinct lengths.

Now we construct the graph G which consists of a number of subgraphs: B_i, $(0 \le i \le 22t, i = 22t + j \ (1 \le j \le \frac{5t-8}{3})$, $i = 23t + \frac{2t-2}{3} + j \ (1 \le j \le \frac{5t-8}{3})$, $i = 32t + j - 60 \ (58 \le j \le t - 742))$.

Now we define these $B_i s$. These subgraphs all only have a common vertex x, otherwise their vertex sets are pairwise disjoint.

For $1 \leq i \leq \frac{5t-8}{3}$, let the subgraph B_{22t+i} consists of a cycle

$$xa_i^1 a_i^2 ... a_i^{28t+\frac{2t-2}{3}+2i-3} x$$

and a path:

$$xa_{i,1}^1 a_{i,1}^2 ... a_{i,1}^{\frac{56t-2}{6}} a_i^{\frac{76t-4}{6}+i}.$$

Based on the construction, B_{22t+i} contains exactly three cycles of lengths:

$$22t+i, 25t+\frac{t-1}{3}+i-1, 28t+\frac{2t-2}{3}+2i-2.$$

For $1 \leq i \leq \frac{5t-8}{3}$, let the subgraph $B_{23t+\frac{2t-2}{3}+i}$ consists of a cycle

$$xb_i^1 b_i^2 ... b_i^{28t+\frac{2t-2}{3}+2i-2} x$$

and a path:

$$xb_{i,1}^1 b_{i,1}^2 ... b_{i,1}^{11t-1} b_i^{\frac{76t-4}{6}+i}.$$

Based on the construction, $B_{23t+\frac{2t-2}{3}+i}$ contains exactly three cycles of lengths:

$$23t + \frac{2t-2}{3} + i, 27t + i - 1, 28t + \frac{2t-2}{3} + 2i - 1.$$

For $58 \leq i \leq t-742$, let the subgraph $B_{32t+i-60}$ consists of a cycle

$$C_{32t+i-60} = xy_i^1 y_i^2 ... y_i^{137t+11i+890} x$$

and ten paths sharing a common vertex x, the other end vertices are on the cycle $C_{32t+i-60}$:

$$xy_{i,1}^{1}y_{i,1}^{2}\cdots y_{i,1}^{11t-2}y_{i}^{21t-59+i}$$

$$xy_{i,2}^{1}y_{i,2}^{2}\cdots y_{i,2}^{12t-2}y_{i}^{31t-53+2i}$$

$$xy_{i,3}^{1}y_{i,3}^{2}\cdots y_{i,3}^{12t-2}y_{i}^{41t+156+3i}$$

$$xy_{i,4}^{1}y_{i,4}^{2}\cdots y_{i,4}^{13t-2}y_{i}^{51t+155+4i}$$

$$xy_{i,5}^{1}y_{i,5}^{2}\cdots y_{i,5}^{13t-2}y_{i}^{61t+155+5i}$$

$$xy_{i,6}^{1}y_{i,6}^{2}\cdots y_{i,6}^{14t-2}y_{i}^{71t+154+6i}$$

$$xy_{i,7}^{1}y_{i,7}^{2}\cdots y_{i,7}^{14t-2}y_{i}^{81t+153+7i}$$

$$xy_{i,8}^{1}y_{i,8}^{2}\cdots y_{i,8}^{15t-2}y_{i}^{91t+147+8i}$$

$$xy_{i,9}^{1}y_{i,9}^{2}\cdots y_{i,9}^{15t-2}y_{i}^{101t+149+9i}$$

$$xy_{i,10}^{1}y_{i,10}^{2}\cdots y_{i,10}^{16t-2}y_{i}^{111t+151+10i}.$$

As a cycle with d chords contains $\binom{d+2}{2}$ distinct cycles, $B_{32t+i-60}$ contains exactly 66 cycles of lengths:

$$
\begin{array}{llll}
32t+i-60, & 33t+i+4, & 34t+i+207, & 35t+i-3, \\
36t+i-2, & 37t+i-3, & 38t+i-3, & 39t+i-8, \\
40t+i, & 41t+i, & 42t+i+739, & 43t+2i-54, \\
43t+2i+213, & 45t+2i+206, & 45t+2i-3, & 47t+2i-3, \\
47t+2i-4, & 49t+2i-9, & 49t+2i-6, & 51t+2i+2, \\
51t+2i+741, & 53t+3i+155, & 54t+3i+212, & 55t+3i+206,
\end{array}
$$

$$
\begin{array}{llll}
56t+3i-4, & 57t+3i-4, & 58t+3i-10, & 59t+3i-7,\\
60t+3i-4, & 61t+3i+743, & 64t+4i+154, & 64t+4i+212,\\
66t+4i+205, & 66t+4i-5, & 68t+4i-10, & 68t+4i-8,\\
70t+4i-5, & 70t+4i+737, & 74t+5i+154, & 75t+5i+211,\\
76t+5i+204, & 77t+5i-11, & 78t+5i-8, & 79t+5i-6,\\
80t+5i+736, & 85t+6i+153, & 85t+6i+210, & 87t+6i+198,
\end{array}
$$

$$\begin{array}{llll}
87t+6i-9, & 89t+6i-6, & 89t+6i+735, & 95t+7i+152, \\
96t+7i+204, & 97t+7i+200, & 98t+7i-7, & 99t+7i+735, \\
106t+8i+146, & 106t+8i+206, & 108t+8i+202, & 108t+8i+734, \\
116t+9i+148, & 117t+9i+208, & 118t+9i+943, & 127t+10i+150, \\
127t+10i+949, & 137t+11i+891. & &
\end{array}$$

B_0 is a path with an end vertex x and length $n - n_t$. The other B_i is simply a cycle of length i.

Then $f(n) \geq n + \frac{119}{3}t - \frac{26399}{3}$, for $n \geq n_t$.

This completes the proof.

Let $f_2(n)$ be the maximum number of edges in a 2-connected graph on n vertices in which no two cycles have the same length.

Y. Shi[On maximum cycle-distributed graphs, Discrete Math. 71(1988) 57-71.] proved that

Theorem 1.8 (Shi 1988) For every integer $n \geq 3$, $f_2(n) \leq n + [\frac{1}{2}(\sqrt{8n-15} - 3)]$.

G. Chen, J. Lehel, M. S.Jacobson, and W. E.Shreve [Note on graphs without repeated cycle lengths, J. Graph Theory, 29(1998),11-15.] proved that

Theorem 1.9 (Chen, Lehel, Jacobson, and Shreve 1998)
$f_2(n) \geq n + \sqrt{n/2} - o(\sqrt{n})$

E. Boros, Y. Caro, Z. Füredi and R. Yuster [Covering non-uniform hypergraphs, J. Combin. Theory Ser. B 82(2001), 270-284.] improved this lower bound significantly:

Theorem 1.10 (Boros, Caro, Füredi and Yuster 2001) $f_2(n) \geq n + \sqrt{n} - O(n^{\frac{9}{20}})$.

Combining this with Shi's results we know:
Corollary 1.11 (Boros, Caro, Füredi and Yuster 2011)

$$\sqrt{2} \geq \limsup \frac{f_2(n) - n}{\sqrt{n}} \geq \liminf \frac{f_2(n) - n}{\sqrt{n}} \geq 1.$$

Boros, Caro, Füredi and Yuster [E. Boros, Y. Caro, Z. Füredi and R. Yuster, Covering non-uniform hypergraphs, J. Combin. Theory Ser. B 82(2001), 270-284.] made the following conjecture:
Conjecture 1.12 (Boros, Caro, Füredi and Yuster 2001) $\lim \frac{f_2(n)-n}{\sqrt{n}} = 1$.

It is easy to see that Conjecture 1.12 implies the (difficult) upper bound in the Erdős Turan Theorem [P. Erdős , On a problem of Sidon in additive number theory and on some related problems. Addendum, J. Lond. Math. Soc. 19 (1944), 208.][P. Erdős and P. Turan, On a problem of Sidon in additive number theory, and on some related problems, J. Lond. Math. Soc. 16 (1941), 212–215.](see [E. Boros, Y. Caro, Z. Füredi and R. Yuster, Covering non-uniform hypergraphs, J. Combin. Theory Ser. B 82(2001), 270-284.]).

Jie Ma, Tianchi Yang [Non-repeated cycle lengths and Sidon sequences，arXiv:2007.12513 , 2020, Israel J. Math., to appear.] prove a conjecture of Boros, Caro, Füredi and Yuster on the maximum number of edges in a 2-connected graph without repeated cycle lengths, which is a restricted version of a longstanding problem of Erdős. Their proof together with the matched lower bound construction of Boros, Caro, Füredi and Yuster show that this problem can be conceptually reduced to the seminal problem of finding the maximum Sidon sequences in number theory.

Theorem 1.13 (Ma, Yang 2020)

Any n-vertex 2-connected graph with no two cycles of the same length contains at most $n + \sqrt{n} + o(\sqrt{n})$ edges.

Let $f_2(n,k)$ be the maximum number of edges in a graph G on n vertices in which no two cycles have the same length and G which consists of k blocks. A natural question is what is the maximum number of edges $f_2(n,k)$. It is clearly that $f_2(n,1) = f_2(n)$.

K. Markström [A note on uniquely pancyclic graphs, Australas. J. Combin. 44 (2009), 105-110.] raised the following problem:

Problem 1.14 (Markström 2009) Determine the maximum number of edges in a hamiltonian graph on n vertices with no repeated cycle lengths.

Let $g(n)$ be the maximum number edges in an n-vertex, Hamiltonian graph with no repeated cycle length. J. Lee, C. Timmons [A note on the number of edges in a Hamiltonian graph with no repeated cycle length, Australas. J. Combin. 69(2)(2017), 286-291.] prove the following.

Theorem 1.15 (Lee and Timmons 2017) If q is a power of a prime and $n = q^2 + q + 1$, then

$$g(n) \geq n + \sqrt{n - 3/4} - 3/2$$

A simple counting argument shows that $g(n) < n + \sqrt{2n} + 1$.

The lower bound $f(0,0,2,\cdots,2)$ is given by J. Xu and Y. Shi [The maximum possible number of edges in a simple graph with at most two cycles having the same lengty, J.Shanghai Normal Univ.(Natural Sciences), 32(3)(2003), 26-32.].

Theorem 1.16 (Xu and Shi 2003)

For $n \geq 3$,

$f(0,0,2,...,2) \geq n-1+[(\sqrt{11n-20})/2]$,

and the equality holds when $3 \leq n \leq 10$.

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turan number.

We denote by $m_i(n)$ the numbers of cycles of length i in the complete graph K_n on n vertices. Obviously,

$$ex(n, C_k)$$

$$= f(0, 0, m_3(n), \cdots,$$

$$m_{k-1}(n), 0, m_{k+1}(n), \cdots, m_n(n))$$

$$= f(0, 0, 2^{\frac{n(n-1)}{2}}, \cdots,$$

$$2^{\frac{n(n-1)}{2}}, 0, 2^{\frac{n(n-1)}{2}}, \cdots, 2^{\frac{n(n-1)}{2}}).$$

Therefore, finding $ex(n, C_k)$ is a special case of determining $f(a_1, a_2, \cdots, a_n)$.

There are not good sufficient and necessary condition of when a graph on n vertices in which no two cycles have the same length. There are also not good sufficient and necessary condition of when a 2-connected graph on n vertices in which no two cycles have the same length. It would be nice to give good sufficient and necessary condition of when a graph on n vertices in which no two cycles have the same length. It would be nice to give good sufficient and necessary condition of when a 2-connected graph on n vertices in which no two cycles have the same length.

The survey article on this problem can be found in

Tian, Feng[The progress on some problems in graph theory. (Chinese) Qufu Shifan Daxue Xuebao Ziran Kexue Ban 1986, no. 2, 30–36. MR0865617 (87m:05160)],

Zhang, Ke Min[Progress of some problems in graph theory. (Chinese) J. Math. Res. Exposition 27 (2007), no. 3, 563–576. MR2349503] and

Chunhui Lai, Mingjing Liu[Some open problems on cycles, Journal of Combinatorial Mathematics and Combinatorial Computing 91 (2014), 51-64.]

Also see Douglas B. West [Introduction to Graph Theory (Second edition), Prentice Hall, 2001] ,p77, Exercises 2.1.41 , Douglas B. West [Combinatorial Mathematics, Cambridge University Press,Cambridge, 2020] ,p249, Exercise 5.4.27 , RMM 2019 - Romanian Master of Mathematics 2019 http://rmms.lbi.ro/rmm2019/index.php?id=home, The 11th Romanian Master of Mathematics Competition Problem 3.

The progress of all 50 problems in [J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, New York, 1976)] can be find in Stephen C. Locke, Unsolved problems: http://math.fau.edu/locke/Unsolved.htm

Outline

P. Erdős conjectured that there exists a positive constant c such that $ex(n, C_{2k}) \geq cn^{1+1/k}$(see P. Erdős, Some unsolved problems in graph theory and combinatorial analysis, Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969) , pp. 97–109, Academic Press, London, 1971).

P. Erdős [Extremal problemes in graph theory, in Theory of graphs and its applications, Proc. Symp. (Smolenice, 1963), (M. Fiedler, ed.), 29-36. New York: Academic Press, 1965.] and J.A Bondy, M. Simonovits [Cycle of even length in graphs, J. Combin. Theory Ser. B,16(1974),97-105.] obtained that

Theorem 2.1 (Erdős 1965, Bondy and Simonovits 1974)

$$ex(n, C_{2k}) \leq ckn^{1+1/k}$$

R. Wenger [Extremal graphs with no $C^{4,}s$,$C^{6,}s$, or $C^{10,}s$,, J. Combin. Theory Ser. B 52(1991), 113-116.] proved that

Theorem 2.2 (Wenger 1991)

$$ex(n, C_4) \geq (\frac{n}{2})^{3/2},$$

$$ex(n, C_6) \geq (\frac{n}{2})^{4/3},$$

$$ex(n, C_{10}) \geq (\frac{n}{2})^{6/5}$$

Z. Furedi[Graph without quadrilaterals, J. Combin. Theory Ser. B 34(1983), 187-190.] proved that

Theorem 2.3 (Furedi 1983) If q is a power of 2, then

$$ex(q^2+q+1, C_4) = q(q+1)^2/2$$

Z. Furedi[On the number of edges of quadrilateral-free graphs, J. Combin. Theory Ser. B 68(1996), 1-6.] proved that

Theorem 2.4 (Furedi 1996) Let G be a quadrilateral-free graph with e edges on q^2+q+1 vertices, and suppose that $q \geq 15$. Then $e \leq q(q+1)^2/2$.

Corollary 2.5 (Furedi 1996) If q is a prime power greater than 13, $n = q^2+q+1$. Then

$$ex(n, C_4) = q(q+1)^2/2.$$

Z. Furedi, A. Naor and J. Verstraete[on the Turan number for the hexagon, Adv. Math. 203(2) (2006), 476-496.] proved that

Theorem 2.6 (Furedi, Naor and Verstraete 2006)

$$ex(n, C_6) > 0.5338n^{4/3}$$

for infinitely many n and

$$ex(n, C_6) < 0.6272n^{4/3}$$

if n is sufficiently large.

This refute the Erdös-Simonovits conjecture in 1982 for hexagons(see[Furedi, Naor and Verstraete 2006]).

Firke, Frank A.; Kosek, Peter M.; Nash, Evan D.; Williford, Jason. [Extremal graphs without 4-cycles. J. Combin. Theory Ser. B 103 (2013), no. 3, 327–336. MR3048158] proved that

Theorem 2.7 (Firke, Kosek, Nash, Williford 2013)

For q even,

$$ex(q^2 + q, C_4) \leq q(q + 1)^2/2 - q.$$

M. Tait, C. Timmons [Sidon sets and graphs without 4-cycles, J. Comb. 5(2) (2014), 155 - 165.] proved that

Theorem 2.8 (M. Tait, C. Timmons 2014)

If q is an odd prime power, then

$$ex(q^2 - q - 2, C_4) \geq \frac{1}{2}q^3 - q^2 - O(q^{\frac{3}{4}}).$$

Jialin He, Jie Ma, Tianchi Yang [Stability and supersaturation of 4-cycles，arXiv:1912.00986v3 [math.CO]] proved that

Theorem 2.9 (He, Ma, Yang 2019)

Let q be even and G be a C_4-free graph on q^2+q+1 vertices with at least $\frac{1}{2}q(q+1)^2 - \frac{1}{2}q + o(q)$ edges. Then there exists a unique polarity graph of order q, which contains G as a subgraph.

The survey article on this Erdos conjecture can be found in

F.R.K,Chung [Open problems of Paul Erdos in graph theory, J. Graph Theory 25 (1997), 3-36.]

Z.Furedi, M. Simonovits [The history of degenerate (bipartite) extremal graph problems. Erdos centennial, 169-264, Bolyai Soc. Math. Stud., 25, Janos Bolyai Math. Soc., Budapest, 2013.]

J. Verstraete [Extremal problems for cycles in graphs, Recent trends in combinatorics, 83-116, IMA Vol. Math. Appl., 159, Springer, Cham, 2016].

Chunhui Lai, Mingjing Liu[Some open problems on cycles, Journal of Combinatorial Mathematics and Combinatorial Computing 91 (2014), 51-64.]

Generalized Turán problems can be found in

N. Alon, R. Yuster [The Turán number of sparse spanning graphs. J. Combin. Theory Ser. B 103(3) (2013), 337 – 343.]

N. Alon, C. Shikhelman [Many T copies in H-free graphs. J. Combin. Theory Ser. B 121 (2016), 146 – 172.]

N. Alon, C. Shikhelman [H-free subgraphs of dense graphs maximizing the number of cliques and their blow-ups. Discrete Math. 342 (2019), 988 – 996.]

P. Loh, M. Tait, C. Timmons, R. M. Zhou [Induced Turán numbers. Combin. Probab. Comput. 27(2) (2018), 274 – 288.]

C. Palmer, M. Tait, C. Timmons, A. Z. Wagner [Turán numbers for Berge-hypergraphs and related extremal problems. Discrete Math. 342 (2019), no. 6, 1553-1563.]

Y. Caro, Z. Tuza[Regular Turán numbers，arXiv:1911.00109 [math.CO]]

Y. Lan, Y. Shi, Z. Song [Extremal theta-free planar graphs. Discrete Math. 342 (2019), no. 12, 111610, 8 pp.]

D. Gerbner, E. Gyoria, A. Methuku, M. Vizer [Generalized Turán problems for even cycles. J. Combin. Theory Ser. B 145 (2020), 169 – 213.]

etc.

There are not good sufficient and necessary condition of when a graph on n vertices in which contains k cycle. For $k = n$, it is Hamiltonnian problem, The survey article on Hamiltonnian problem can be found in [Gould, Ronald J. Updating the Hamiltonian problem—a survey. J. Graph Theory 15 (1991), no. 2, 121–157. MR1106528 (92m:05128); Gould, Ronald J. Advances on the Hamiltonian problem—a survey. Graphs Combin. 19 (2003), no. 1, 7–52. MR1974368 (2004a:05092); Gould, Ronald J. Recent advances on the Hamiltonian problem: Survey III. Graphs Combin. 30 (2014), no. 1, 1–46. MR3143857 (Reviewed)]. It would be nice to give good sufficient and necessary condition of when a graph on n vertices in which contains k cycle for some k.

Outline

In 1973, R. C. Entringer raised the problem of determine which simple graphs G have exactly one cycle of each length l, $3 \leq l \leq v$ (see J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, New York, 1976), p.247, Problem 10). This graph is called a uniquely pancyclic graph.

A graph G is outerplanar If it has a planar embedding $\widetilde{G}$ in which all vertices lie on the boundary of its outer face.

Y. Shi[Some theorems of uniquely pancyclic graphs, Discrete Math. 59, 167-180 (1986).Zbl 0589.05046] proved that

Theorem 3.1 (Shi 1986) A graph G is an outerplanar uniquely pancyclic graph if and only if $G \in \{K_3, G_5, G_8^{(1)}, G_8^{(2)}\}$.

Theorem 3.2 (Shi 1986) A graph G with $v+m$ edges for $m \leq 3$ is a uniquely pancyclic graph if and only if
$G \in \{K_3, G_5, G_8^{(1)}, G_8^{(2)}, G_{14}^{(1)}, G_{14}^{(2)}, G_{14}^{(3)}\}$.
Theorem 3.3 (Shi 1986) None of the graphs each of which contains $v+4$ edges is a uniquely pancyclic graph.

and proposed the following conjecture:

Conjecture 3.4 (Shi 1986) None of the graphs each of which contains $v+m$ edges for $m \geq 4$ is a uniquely pancyclic graph.

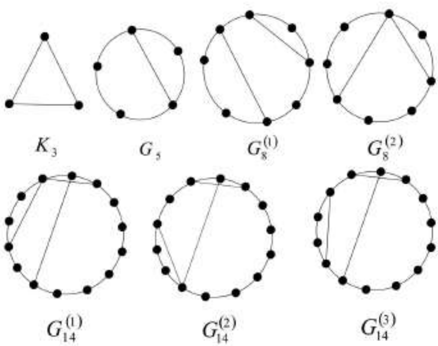

Figure: uniquely pancyclic graphs.

Y. Shi, H.P. Yap, S.K. Teo [On uniquely r-pancyclic graphs, (English) Capobianco, Michael F. (ed.) et al., Graph theory and its applications: East and West. Proceedings of the first China-USA international conference, held in Jinan, China, June 9-20, 1986. New York: New York Academy of Sciences,. Ann. N. Y. Acad. Sci. 576, 487-499 (1989).], Y. Shi, L. Xu, X. Chen, M. Wang[Almost uniquely pancyclic graphs, Adv. Math. (China) 35 (2006), no. 5, 563–569.], Y.Shi[A class of almost uniquely pancyclic graphs, J. Systems Sci. Math. Sci. 26 (2006), no. 4, 433–439.], Y. Bu [The enumeration of uniquely pancyclic digraph with the least arcs, Math. Pract. Theory 39, No. 4, 189-193(2009)],J. Chen[A Note on r-$(P_0, \cdots, P_{t-1})$-pancyclic graphs, J. Minnan Norm. Univ., Nat. Sci. 28(4)(2015), 9-19.], Y. Zhang[Some new results on r-$(d_1, d_2, \cdots, d_{t-1})$-pancyclic Graphs] obtained some results.

K. Markstrom[A note on uniquely pancyclic graphs, Australas. J. Comb. 44, 105-110 (2009)] consider uniquely pancyclic graphs(UPC graph), i.e. n vertex graphs with exactly one cycle of each length from 3 to n. The first result of the paper gives new upper and lower bounds on the number of edges in a uniquely pancyclic graph.

Since a UPC graph G on n vertices must be hamiltonian it can be constructed by adding chords to a cycle C_n of length n, let k denote the number of chords added. We say that two chords e_1 and e_2 cross each other if $C_n \cup e_1 \cup e_2$ is a K_4-subdivision. Let c denote the number of crossing pairs of chords, let c_3 denote the number of unordered triples of pairwise crossing chords. Let $c_\triangle$ be the number of unordered triples $\{e_1, e_2, e_3\}$ of chords such that no pair of chords e_i, e_j cross each other and there is a cycle using all three chords.

Theorem 3.5 (Markstrom 2009) The number of chords k in a UPC graph G satisfy inequality (1), and if G has at least 5 chords inequality (2) as well:

$$3+2k+\binom{k}{2}+(k-1)c-c_3+c_\triangle \leq n \tag{1}$$

$$\log_2(n-1+f(k,\triangle))+\log_2(\frac{4}{7}) \leq k, \text{for } k \geq 4 \tag{2}$$

with

$$f(k,\triangle)=\begin{cases} 2\binom{k-2}{3}, & \text{for } \triangle=3 \\ 2^{k-2}7(1-\frac{8}{7}(1+\triangle+\frac{1}{2}\binom{\triangle}{2}))), & \text{for } \triangle\geq 4 \end{cases}$$

Next he report on a computer search for new uniquely pancyclic graphs. He found that there are no new such graphs on $n \leq 59$ vertices and that there are no uniquely pancyclic graphs with exactly 5 chords.

C. T. Zamfirescu [(2)-pancyclic graphs, Discrete Applied Mathematics, 161(2013),1128-1136.] introduce the class of (2)-pancyclic graphs, which are simple undirected finite connected graphs of order n having exactly two cycles of length p for each p satisfying $3 \leq p \leq n$, analyze their properties, and give several examples of such graphs (show in the following figure), among which are the smallest.

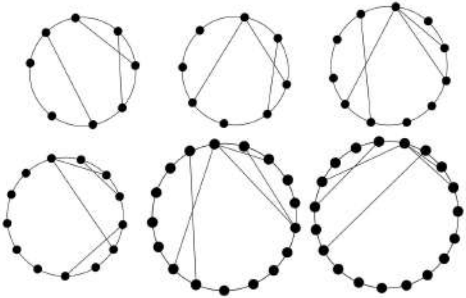

Figure: several examples of (2)-pancyclic graphs.

Theorem 3.6 (Zamfirescu 2013) Every (2)-pancyclic graph has minimal degree 2 and maximal degree at most $\lfloor \frac{\sqrt{16n+1}-3}{2} \rfloor$.

Theorem 3.7 (Zamfirescu 2013) An edge in a (2)-pancyclic graph lies on at least $\lceil \log_2(2n-3) \rceil + 1$ and at most $2n - \lceil \log_2(2n-3) \rceil - 4$ cycles.

The following conjecture and questions arose in C. T. Zamfirescu [(2)-pancyclic graphs, Discrete Applied Mathematics, 161(2013),1128-1136.]

Conjecture 3.8 (Zamfirescu 2013) There are finitely many (2)-pancyclic graphs.

A graph with v vertices is (r)-pancyclic if it contains precisely r cycles of every length from 3 to v. A bipartite graph with even number of vertices v is said to be (r)-bipancyclic if it contains precisely r cycles of each even length from 4 to v. A bipartite graph with odd number of vertices v and minimum degree at least 2 is said to be oddly (r)- bipancyclic if it contains precisely r cycles of each even length from 4 to $v-1$.

Problem 3.9 (Zamfirescu 2013) Is every (2)-pancyclic graph planar?

Problem 3.10 (Zamfirescu 2013) Do (k)-pancyclic graphs exist for $k \geq 3$?

Problem 3.11 (Zamfirescu 2013) For which $k \geq 2$ do infinite families of (k)-pancyclic oriented graphs (or digraphs) exist?

Let n, r, k be positive integers where $3 \le r \le n$ and $k \ge 2$. A graph G of order n is said to be r-(k)-*pancyclic* if G contains exactly k t-cycles for all t satisfying $r \le t \le n$. S. Liu[On r-(k)-pancyclic graphs. Ars Combin. 140 (2018), 277–291. MR3822005] proved that

Theorem 3.12 (Liu 2018) Let $2^{\lambda-\mu} + \lambda - 1 \le n < 2^{\lambda-\mu+1} + \lambda$ and

$$r_{(n,\lambda,\mu)} = \begin{cases} 2^{\lambda-\mu-1} + 3, & \text{if } n \le 3 \cdot 2^{\lambda-\mu-1} + 2; \\ n - 2^{\lambda-\mu} + 1, & \text{if } n > 3 \cdot 2^{\lambda-\mu-1} + 2. \end{cases}$$

Then there exist $r_{(n,\lambda,\mu)}$-(2^{μ})-pancyclic graphs of order n.

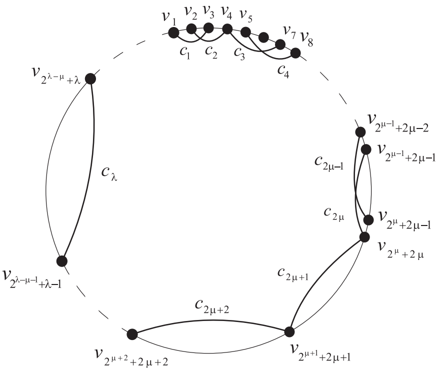

Figure: r-(2^μ)-pancyclic graphs of order n.

Abdollah Khodkar, Oliver Sawin, Lisa Mueller, WonHyuk Choi, [(r)-Pancyclic, (r)-Bipancyclic and Oddly (r)-Bipancyclic Graphs,arXiv:1510.03052] using computer search classify all (r)-Pancyclic and (r)-Bipancyclic graphs with v vertices and at most $v+5$ edges. They also classify all Oddly (r)-Bipancyclic Graphs with v vertices and at most $v+4$ edges.

The survey article on this problem can be found in

Tian, Feng[The progress on some problems in graph theory. (Chinese) Qufu Shifan Daxue Xuebao Ziran Kexue Ban 1986, no. 2, 30–36. MR0865617 (87m:05160)] and

Zhang, Ke Min[Progress of some problems in graph theory. (Chinese) J. Math. Res. Exposition 27 (2007), no. 3, 563–576. MR2349503]

George John C., Khodkar Abdollah, Wallis W. D., [Pancyclic and bipancyclic graphs, Springer Briefs in Mathematics, Springer, Cham, 2016. MR3495006]

Outline

1. Erdős Problem
2. Erdős conjecture
3. Entringer problem
4. Hajós conjecture

Hajós conjecture

An *eulerian graph* is a graph (not necessarily connected) in which each vertex has even degree. Let G be an eulerian graph. A *circuit decomposition* of G is a set of edge-disjoint circuits $C_1, C_2, \cdots, C_t$ such that $E(G) = C_1 \cup C_2 \cup \cdots \cup C_t$. It is well known that every eulerian graph has a circuit decomposition. A natural question is to find the smallest number t such that G has a circuit decomposition of t circuits?

Such smallest number t is called the *circuit decomposition number* of G, denoted by $cd(G)$. For each edge $xy \in E(G)$, let $m(xy)$ be the number of edges between x and y. The *multiple number* of G is defined by $m(G) = \sum\limits_{uv \in E(G)} (m(uv) - 1)$. (See G. Fan and B. Xu [Hajós conjecture and projective graphs. Discrete Math. 252 (2002), no. 1-3, 91 - 101])

The following conjecture is due to Hajós(see L. Lovasz, On covering of graphs, in: P. Erdos, G.O.H. Katona (Eds.), Theory of Graphs, Academic Press, New York, 1968, pp. 231 - 236).

Hajós conjecture: $cd(G) \leq \frac{|V(G)|}{2}$ for every simple eulerian graph G.

N. Dean [What is the smallest number of dicycles in a dicycle decomposition of an Eulerian digraph? J. Graph Theory 10 (1986), no. 3, 299–308.] proved that

Theorem 4.1 (Dean 1986) Hajós conjecture is equivalent to the following statement: If G is even, then $cd(G) \leq \frac{|V(G)|-1}{2}$.

L. Lovasz [On covering of graphs, in: P. Erdos, G.O.H. Katona (Eds.), Theory of Graphs, Academic Press, New York, 1968, pp. 231 - 236] proved that

Theorem 4.2 (Lovasz 1968) A graph of n vertices can be covered by $\leq [n/2]$ disjoint paths and circuits.

T. Jiang [On Hajós conjecture, J. China Univ. Sci. Tech. 14 (1984) 585 - 592 (in Chinese).] and K. Seyffarth [Hajós conjecture and small cycle double covers of planar graphs, Discrete Math. 101 (1992) 291 - 306.] proved that

Theorem 4.3 (Jiang 1984; Seyffarth 1992) $cd(G) \leq \frac{|V(G)|-1}{2}$ *for every simple planar eulerian graph* G.

A. Granville, A. Moisiadis [On Hajós conjecture, in: Proceedings of the 16th Manitoba Conference on Numerical Mathematics and Computing, Congr. Numer. 56 (1987) 183 - 187.] and O. Favaron, M. Kouider [Path partitions and cycle partitions of eulerian graphs of maximum degree 4, Studia Sci. Math. Hungar. 23 (1988) 237 - 244.] proved that

Theorem 4.4 (Granville and Moisiadis 1987; Favaron and Kouider 1988) If G is an even multigraph of order n, of size m, with $\Delta(G) \leq 4$, then $cd(G) \leq \frac{n+M-1}{2}$ *where* $M = m - m^*$ *and* m^* *is the size of the simple graph induced by* G.

G. Fan and B. Xu [Hajós conjecture and projective graphs. Discrete Math. 252 (2002), no. 1-3, 91 - 101] proved that

Theorem 4.5 (Fan and Xu 2002) *If G is an eulerian graph with*

$$cd(G) > \frac{|V(G)| + m(G) - 1}{2},$$

then G has a reduction H such that

$$cd(H) > \frac{|V(H)| + m(H) - 1}{2}$$

and the number of vertices of degree less than six in H plus $m(H)$ is at most one.

Corollary 4.6 (Fan and Xu 2002) *Hajós conjecture is valid for projective graphs.*
Corollary 4.7 (Fan and Xu 2002) *Hajós conjecture is valid for K_6^- minor free graphs.*

B. Xu [Hajós conjecture and connectivity of Eulerian graphs.J. Syst. Sci. Complex. 15 (2002), no. 3, 295 - 298.] also proved the following two results:

Theorem 4.8 (Xu 2002) *If G is an eulerian graph with*

$$cd(G) > \frac{|V(G)| + m(G) - 1}{2}$$

such that

$$cd(H) \leq \frac{|V(H)| + m(H) - 1}{2}$$

for each proper reduction of G, then G is 3-connected. Moreover, if $S = \{x, y, z\}$ is a 3-cut of G, letting G_1 and G_2 be the two induced subgraph of G such that $V(G_1) \cap V(G_2) = S$ and $E(G_1) \cup E(G_2) = E(G)$, then either S is not an independent set, or G_1 and G_2 are both eulerian graphs.

Corollary 4.9 (Xu 2002) *To prove Hajós' conjecture, it suffices to prove*

$$cd(G) \leq \frac{|V(G)| + m(G) - 1}{2}$$

for every 3-connected eulerian graph G.

G. Fan [Covers of Eulerian graphs. J. Combin. Theory Ser. B 89 (2003), no. 2, 173 - 187.] proved that

Theorem 4.10 (Fan 2003) Every eulerian graph on n vertices can be covered by at most $\lfloor \frac{n-1}{2} \rfloor$ circuits such that each edge is covered an odd number of times.

This settles a conjecture made by Chung in 1980(see[Fan 2003]).

B. Xu and L. Wang [Decomposing toroidal graphs into circuits and edges. Discrete Appl. Math. 148 (2005), no. 2, 147 - 159.] give

Theorem 4.11 (Xu and Wang 2005) *The edge set of each even toroidal graph can be decomposed into at most* $(n+3)/2$ *circuits in* $O(mn)$ *time, where a toroidal graph is a graph embedable on the torus.*

Theorem 4.12 (Xu and Wang 2005) *The edge set of each toroidal graph can be decomposed into at most* $3(n-1)/2$ *circuits and edges in* $O(mn)$ *time.*

E. Fuchs, L. Gellert, I. Heinrich [Cycle decompositions of pathwidth-6 graphs，Journal of Graph Theory, 94(2)2020, 224-251] give

Theorem 4.13 (Fuchs, Gellert, Heinrich 2020) Every Eulerian graph G of pathwidth at most 6 satisfies Hajós conjecture.

We do not think Hajós conjecture is true.

By the proof of Lemma 3.3 in N. Dean [What is the smallest number of dicycles in a dicycle decomposition of an Eulerian digraph? J. Graph Theory 10 (1986), no. 3, 299–308], if exists k vertices counterexample, then will exist $ik - i + 1 (i = 2, 3, 4, ...)$ vertices counterexamples.

A related problem is as follows Gallai's conjecture(see [L. Lovasz, On covering of graphs, in: P. Erdos, G.O.H. Katona (Eds.), Theory of Graphs, Academic Press, New York, 1968, pp. 231 - 236]):
Conjecture 4.14 (Gallai's conjecture) every simple connected graph on n vertices can be decomposed into at most $(n+1)/2$ paths.

L. Lovasz [On covering of graphs, in: P. Erdos, G.O.H. Katona (Eds.), Theory of Graphs, Academic Press, New York, 1968, pp. 231 - 236] proved that

Theorem 4.15 (Lovasz 1968) If a graph has u odd vertices and g even vertices ($g \geq 1$). then it can be covered by $u/2 + g - 1$ disjoint paths.

Theorem 4.16 (Lovasz 1968) Let a locally finite graph have only odd vertices. Then it can be covered by disjoint finite paths so that every vertex is the endpoint of just one covering path.

The path number of a graph G, denoted $p(G)$, is the minimum number of edge-disjoint paths covering the edges of G. A. Donald[An upper bound for the path number of a graph. J. Graph Theory 4 (1980), no. 2, 189–201.] proved that

Theorem 4.17 (Donald 1980) If a graph with u vertices of odd degree and g nonisolated vertices of even degree. Then

$$p(G) \leq u/2 + [\frac{3}{4}g] \leq [\frac{3}{4}n].$$

L. Pyber [An Erdos-Gallai conjecture. Combinatorica 5 (1985), no. 1, 67–79.] proved that

Theorem 4.18 (Pyber 1985) A graph G of n vertices can be covered by $n-1$ circuits and edges.

Theorem 4.19 (Pyber 1985) Let G be a graph of n vertices and $\{C_1, \ldots, C_k\}$ be a set of circuits and edges such that $\cup_{i=1}^{k} E(C_i) = E(G)$ and k is minimal. Then we can choose k different edges, $e_i \in E(C_i)$, such that these edges form a forest in G.
Theorem 4.20 (Pyber 1985) Let G be a graph of n vertices not containing C_4. Then G can be covered by $[(n-1)/2]$ circuits and $n-1$ edges.

L. Pyber [Covering the edges of a connected graph by paths. J. Combin. Theory Ser. B 66 (1996), no. 1, 152–159.] proved that

Theorem 4.21 (Pyber 1996) Every connected graph G on n vertices can be covered by $n/2 + o(n^{3/4})$ paths.

Theorem 4.22 (Pyber 1996) Every connected graph on n vertices with e edges can be covered by $n/2 + 4(e/n)$ paths.

B. Reed [Paths, stars and the number three. Combin. Probab. Comput. 5 (1996), no. 3, 277–295.] proved that

Theorem 4.23 (Reed 1996) Any connected cubic graph G of order n can be covered by $\lceil n/9 \rceil$ vertex disjoint paths.

N. Dean, M. Kouider [Gallai's conjecture for disconnected graphs. Selected topics in discrete mathematics (Warsaw, 1996). Discrete Math. 213 (2000), no. 1-3, 43–54.] and L. Yan, [On path decompositions of graphs, Thesis (Ph.D.), Arizona State University. ProQuest LLC, Ann Arbor, MI, 1998.] proved that

Theorem 4.24 (Dean, Kouider 2000 and Yan 1998) If a graph(possibly disconnected) with u vertices of odd degree and g nonisolated vertices of even degree. Then

$$p(G) \leq u/2 + [\frac{2}{3}g].$$

G. Fan [Subgraph coverings and edge switchings. J. Combin. Theory Ser. B 84 (2002), no. 1, 54–83.] proved that

Theorem 4.25 (Fan 2002) Every connected graph on n vertices can be covered by at most $\lceil n/2 \rceil$ paths.

This settles a conjecture made by Chung in 1980(see[Fan 2002]).

Theorem 4.26 (Fan 2002) Every 2-connected graph on n vertices can be covered by at most $\lfloor \frac{2n-1}{3} \rfloor$ circuits.

This settles a conjecture made by Bondy in 1990(see[Fan 2002]).

Theorem 4.27 (Fan 2002) Let G be a 2-connected graph on n vertices. Then G can be covered by at most $\lfloor \frac{3(n-1)}{4} \rfloor$ circuits.

G. Fan [Path decompositions and Gallai's conjecture. J. Combin. Theory Ser. B 93 (2005), no. 2, 117–125.] define a graph operation, called α-operation and proved that

Definition 4.28(Fan 2005) Let H be a graph. A pair (S, y), consisting of an independent set S and a vertex $y \in S$, is called an α-pair if the following holds: for every vertex $v \in S\ y$, if $d_H(v) \geq 2$, then (a) $d_H(u) \leq 3$ for all $u \in N_H(v)$ and (b) $d_H(u) = 3$ for at most two vertices $u \in N_H(v)$. (That is, all the neighbors of v has degree at most 3, at most two of which has degree exactly 3.) An α-operation on H is either (i) add an isolated vertex or (ii) pick an α-pair (S, y) and add a vertex x joined to each vertex of S, in which case the ordered triple (x, S, y) is called the α-triple of the α-operation.

Definition 4.29(Fan 2005) An α-graph is a graph that can be obtained from the empty set via a sequence of α-operations.

Theorem 4.30 (Fan 2005) Let G be a graph on n vertices (not necessarily connected). If the E-subgraph of G is an α-graph, then G can be decomposed into $\lfloor n/2 \rfloor$ paths.

Theorem 4.31 (Fan 2005) Let G be a graph on n vertices (not necessarily connected). If each block of the E-subgraph of G is a triangle-free graph of maximum degree at most 3, then G can be decomposed into $\lfloor n/2 \rfloor$ paths.

P.Harding and S. McGuinness [Gallai's conjecture for graphs of girth at least four. J. Graph Theory 75 (2014), no. 3, 256–274. MR3153120] proved that

Theorem 4.32 (Harding and McGuinness 2014) For every simple graph G have girth $g \geq 4$, with u vertices of odd degree and w nonisolated vertices of even degree, there is a path-decomposition having at most $u/2 + \lfloor \frac{g+1}{2g} w \rfloor$ paths.

Botler, F.; Jiménez, A.; Sambinelli, M. [Gallai's path decomposition conjecture for triangle-free planar graphs. Discrete Math. 342(2019), no. 5, 1403-1414.] proved that

Theorem 4.33 (Botler, Jiménez, Sambinelli 2019) Every triangle-free planar graph is a Gallai graph.

Fάbio Botler，Maycon Sambinelli，Rafael S. Coelho，Orlando Lee[Gallai's path decomposition conjecture for graphs with treewidth at most 3，Journal of Graph Theory, 93（3）（2020），328-349.] verify Gallai's conjecture for graphs with treewidth at most 3 and proved that

Theorem 4.34 (Botler，Sambinelli，Coelho，Lee 2020) Every connected planar graph with girth at least 6 is a Gallai graph.

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The End

Thanks for your attention!